1. A bead of mass \( m \) slides without friction on a rotating wire hoop of radius \( a \) whose axis of rotation is through a vertical diameter as shown in Fig. 2.17. The constant angular velocity of the hoop is \( \omega \).

(a) Write the Lagrangian for the system and find any constants of the motion that may exist.

(b) Locate the positions of equilibrium of the bead for \( \omega < \omega_c \) and \( \omega > \omega_c \), where

\[
\omega_c = \sqrt{\frac{g}{a}}.
\]

Which of these positions of equilibrium are stable and unstable?

(c) Calculate the oscillation frequencies of small amplitude vibrations about the points of stable equilibrium.

![Fig. 2.17.](image)

2. (a) Two men are holding opposite ends of a long, uniform, horizontal plank. One man drops his end. Show that, at that moment, the vertical force exerted by the plank on the other man is thereby reduced by a factor of two.

(b) Consider a disk of mass \( m \) sliding on a smooth surface. It oscillates in simple harmonic motion in a straight line under the influence of two equal springs, each of spring constant \( K \) and negligible mass.

Find the effect on the amplitude and frequency of the motion if a piece of putty, also of mass \( m \), is dropped vertically (and remains) on the disk:

1. when the disk is at the extreme end of its oscillation.
2. when the disk is at the center of its oscillation.
3. A motor turns a vertical shaft to which is attached a simple pendulum of length \( l \) and mass \( m \) as shown in Fig. 2.72. The pendulum is constrained to move in a plane. This plane is rotated at constant angular speed \( \omega \) by the motor.
   (a) Find the equations of motion of the mass \( m \).
   (b) Solve the equations of motion, obtaining the position of the mass as a function of time for all possible motions of this system. For this part use small angle approximations.
   (c) Find the angular frequencies of any oscillatory motions.
   (d) Find an expression for the torque that the motor must supply.
   (e) Is the total energy of this system constant in time? Is the Hamiltonian function constant in time? Explain briefly.

![Fig. 2.72.](image)

4. A spherical pendulum consists of a point mass \( m \) tied by a string of length \( l \) to a fixed point, so that it is constrained to move on a spherical surface as shown in Fig.2.14.

![Fig. 2.14.](image)

(a) With what angular velocity will it move on a circle, with the string making a constant angle \( \theta_0 \) with the vertical?
(b) The mass in the circular orbit as in part (a) above receives an impulse
perpendicular to its velocity, resulting in an orbit which has its highest point with the string making an angle $\theta_1$ with the vertical. Write down (but do not try to solve) the equation which may be solved for the angle the string makes with the vertical when the mass is at its lowest point.

(c) For the case in which the amplitude of the oscillations about $\theta_0$ is small, solve for the frequency of these oscillations.

5. A nonrelativistic electron of mass $m$, charge $-e$ in a cylindrical magneton moves between a wire of radius $a$ at a negative electric potential $-\phi_0$ and a concentric cylindrical conductor of radius $R$ at zero potential. There is a uniform constant magnetic field $B$ parallel to the axis. Use cylindrical coordinates $r, \theta, z$.

The electric and magnetic vector potentials can be written as

\[ \phi = -\phi_0 \frac{\ln(r/R)}{\ln(a/R)} \]
\[ A = \frac{1}{2} B r e_\theta \]

($e_\theta$ is a unit vector in the direction of increasing $\theta$)

(a) Write the Lagrangian and Hamiltonian functions.

(b) Show that there are three constants of the motion. Write them down, and discuss the kinds of motion which can occur.

(c) Assuming that an electron leaves the inner wire with zero initial velocity, there is a value of the magnetic field $B_c$ such that for $B \leq B_c$ the electron can reach the outer cylinder, and for $B > B_c$ the electron cannot reach the outer cylinder. Find $B_c$ and make a sketch of the electron’s trajectory for this case. You may assume that $R >> a$.

6. A particle under the action of gravity slides on the inside of a smooth paraboloid whose axis is vertical. Using the distance from the axis, $r$, and the azimuthal angle $\phi$ as generalized coordinates, find

(a) The Lagrangian of the system.

(b) The generalized momenta and the corresponding Hamiltonian.

(c) The equation of motion for the coordinate $r$ as a function of time.

(d) If $\frac{d\phi}{dt} = 0$, show that the particle can execute small oscillations about the lowest point of the paraboloid, and find the frequency of these oscillations.

7. A particle is attracted to a force center by a force which varies inverse as the cube of its distance from the center. Derive the equations of motion and solve them for the
orbits. Discuss how the nature of the orbits depends on the parameters of the system.

8. Consider a hoop of radius $a$ in a vertical plane rotating with angular velocity $\omega$ about a vertical diameter. Consider a bead of mass $m$ which slides without friction on the hoop as indicated in Fig. 1.69.

(a) Under what specific condition will the equilibrium of the bead at $\theta = 0$ be stable?
(b) Find another value of $\theta$ for which, in certain circumstances, the bead will be in stable equilibrium. Indicate the value of $\omega$ for which this stable equilibrium takes place.
(c) Explain your answer with the aid of appropriate graphs of the potential energy versus $\theta$ as measured in the rotating frame.

![Fig. 1.69.](image)

9. The classical interaction between two inset gas atoms, each of mass $m$, is given by the potential

$$V(r) = \frac{2A}{r^6} + \frac{B}{r^2}, \quad A, B > 0, \quad r = |r_1 - r_2|.$$

(a) Give the Hamiltonian for the system of the two atoms.
(b) Describe completely the lowest energy classical state(s) of this system.
(c) If the energy is slightly higher than the lowest [part(b)], what are the possible frequencies of the motion of the system?

10. A thin, uniform rod of length $2L$ and mass $M$ is suspended from a massless string of length $l$ tied to a nail. As shown in Fig. 2.9, a horizontal force $F$ is applied to the rod’s free end.

Write the Lagrange equations for this system. For very short times (so that all angles are small) determine the angles that the string and the rod make with the vertical. Start from rest at $t=0$. Draw a diagram to illustrate the initial motion of the rod.
11. (a) Show that the moment of inertia of a thin rod about its center of mass is \( \frac{ml^2}{12} \).

(B) A long thin tube of negligible mass is pivoted so that it may rotate without friction in a horizontal plane. A thin rod of mass \( M \) and length \( l \) slides without friction in the tube. Choose a suitable set of coordinates and write Lagrange’s equations for this system.

(c) Initially the rod is centered over the pivot and the tube is rotating with angular velocity \( \omega_0 \). Show that the rod is unstable in this position, and describe its subsequent motion if it is disturbed slightly. What are the radial and angular velocities of the rod after a long time? (Assume the tube is long enough that the rod is still inside.)

12. A rigid wheel has principal moments of inertia \( I_1 = I_2 \neq I_3 \) about its body-fixed principal axes \( \hat{x}_1, \hat{x}_2 \) and \( \hat{x}_3 \), as shown in Fig. 1.179. The wheel is attached at its center of mass to a bearing which allows frictionless rotation about one space-fixed axis. The wheel is “dynamically balanced”, i.e. it can rotate at constant \( \omega \neq 0 \) and exert no torque on its bearing. What conditions must the components of \( \omega \) satisfy? Sketch the permitted motion(s).
13. Three identical objects, each of mass \( m \), are connected by springs of spring constant \( K \), as shown in Fig. 1.95. The motion is confined to one dimension.

At \( t=0 \), the masses are at rest at their equilibrium positions. Mass A is then subjected to an external time-dependent driving force \( F(t) = f \cos(\omega t) \), \( t>0 \). Calculate the motion of mass \( C \).

![Fig. 1.95](image)

14. A simple classical model of the \( CO_2 \) molecule would be a linear structure of three masses with the electrical forces between the ions represented by two identical springs of equilibrium length \( l \) and force constant \( k \), as shown in Fig. 1.90. Assume that only motion along the original equilibrium line is possible, i.e., ignore rotations. Let \( m \) be the mass of \( O^- \) and \( M \) be the mass of \( C^{++} \).

(a) How many vibrational degrees of freedom does this system have?
(b) Define suitable coordinates and determine the equation of motion of the masses.
(c) Seek a solution to the equations of motion in which all particles oscillate with a common frequency (normal modes) and calculate the possible frequencies.
(d) Calculate the relative amplitudes of the displacements of the particles for each of these modes and describe the nature of the motion for each mode. You may use a sketch as part of your description.
(e) Which modes would you expect to radiate electromagnetically and what is the multipole order of each?

![Fig. 1.90](image)

15. A small homogeneous sphere of mass \( m \) and radius \( r \) rolls without sliding on the outer surface of a larger stationary sphere of radius \( R \) as shown in Fig. 1.165. Let \( \theta \) be the polar angle of the small sphere with respect to a coordinate system with origin at the center of the large sphere and \( z \)-axis vertical. The smaller sphere starts from rest at the top the larger sphere (\( \theta=0 \)).

(a) Calculate the velocity of the center of the small sphere as a function of \( \theta \).
(b) Calculate the angle at which the small sphere flies off the large one.
(c) If one now allows for sliding with a coefficient of friction \( \mu \), at what point will the small sphere start to slide?

![Image](image_url)

Fig. 1.165.

16. A perfectly smooth horizontal disk is rotating with an angular velocity \( \omega \) about a vertical axis passing through its center. A person on the disk at a distance \( R \) from the origin gives a perfectly smooth coin (negligible size) of mass \( m \) a push toward the origin. This push gives it an initial velocity \( V \) relative to the disk. Show that the motion for a time \( t \), which is such that \( (\omega t)^2 \) is negligible, appears to the person on the disk to be a parabola, and give the equation of the parabola.

17. Four masses, all of value \( m \), lie in the \( xy \)-plane at positions \((x, y) = (a, 0), (-a, 0), (0, +2a), (0, -2a)\). These are joined by massless rods to form a rigid body.
(a) Find the inertial tensor, using the \( x \)-, \( y \)-, \( z \)-axes as reference system. Exhibit the tensor as a matrix.
(b) Consider a direction given by unit vector \( \hat{n} \) that lies “equally between” the positive \( x \)-, \( y \)-, \( z \)-axes, i.e. it makes equal angels with these three directions. Find the moment of inertia for rotation about this axis.
(c) Given that at a certain time \( t \) the angular velocity vector lies along the above direction \( \hat{n} \), find, for that instant, the angle between the angular momentum vector and \( \hat{n} \).

18. (a) Let us apply a shearing force on a rectangular solid block as shown in Fig. 2.77. Find the relation between the displacement \( u \) and the applied force within elastic limits.
(b) The elastic properties of a solid support elastic waves. Assume a transverse plane wave which proceeds in the \( x \)-direction and whose oscillations are in the \( y \)-direction. Derive the equations of motion for the displacement.
(c) Find the expression for the speed of the transverse elastic wave.
19. A particle is constrained to be in a plane. It is attracted to a fixed point \( P \) in this plane; the force is always directed exactly at \( P \) and is inversely proportional to the square of the distance from \( P \).
   (a) Using polar coordinates, write the Lagrangian of this particle.
   (b) Write Lagrangian equations for this particle and find at least one first integral.

20. Two mass points \( m_1 \) and \( m_2 \) \((m_1 \neq m_2)\) are connected by a string of length \( l \) passing through a hole in a horizontal table. The string and mass points move without friction with \( m_1 \) on the table and \( m_2 \) free to move in a vertical line.
   (a) What initial velocity must \( m_1 \) be given so that \( m_2 \) will remain motionless a distance \( d \) below the surface of the table?
   (b) If \( m_2 \) is slightly displaced in a vertical direction, small oscillations ensue. Use Lagrange’s equations to find the period of these oscillations.

21. A communication satellite is shot into an elliptical orbit about the earth. The orbit has apogee \( r_1 \) and perigee \( r_2 \). The satellite has mass \( m \), the earth has mass \( M \) and the gravitational constant is \( G \) (assume \( m/M<<1 \)). For effective transmission the satellite must be put into a circular orbit. What velocity increment (direction and magnitude) must be given to the satellite to put it into a circular orbit of radius \( r_2 \)? Where on the orbit is the velocity kick applied?

22. A bead of mass \( m \) is free to slide along a smooth rigid circular wire which is forced to rotate with a constant angular velocity about the vertical diameter. Consider a small displacement from the position (not on axis of rotation) of stable equilibrium
for an angular velocity $\omega$ and obtain the condition for, and frequency of simple harmonic motion.

23. A square lamina, slide $2a$, mass $m$, is lying on a table when struck on a corner by a bullet of mass $m$ with velocity $v$ parallel to one of the edges of the lamina (inelastic). Find the subsequent angular velocity.

24. The transformation equations between two sets of coordinates are

$$Q = \ln(1 + \frac{1}{q^2} \cos p)$$

$$P = 2(1 + \frac{1}{q^2} \cos p)q^2 \sin p$$

(a) show directly from these transformation equations that $Q, P$ are canonical variables if $q$ and $p$ are.

(b) Show that the function that generates this transformation between the two sets of canonical variables is

$$F_3 = -[\exp(Q) - 1]^2 \tan p$$

25. Three identical cylinders rotate with the same angular velocity $\Omega$ about parallel central axes. They are brought together until they touch, keeping the axes parallel. A new steady state is achieved when, at each contact line, a cylinder does not slip with respect to its neighbor as shown in Fig.1.129. How much of the original spin kinetic energy is now left?

(The precise order in which the first and second touch, and the second and third touch, is irrelevant.)
26. Consider a particle of mass $m$ which is constrained to move on the surface of a sphere of radius $R$. There are no external forces of any kind on the particle.
(a) What is the number of generalized coordinates necessary to describe the problem?
(b) Choose a set of generalized coordinates and write the Lagrangian of the system.
(c) What is the Hamiltonian of the system? Is it conserved?
(d) Prove that the motion of the particle is along a great circle of the sphere.

27. Consider a solid cylinder of mass $m$ and radius $r$ sliding without rolling down the smooth inclined face of a wedge of mass $M$ that is free to move on a horizontal plane without friction (Fig. 1.174)
(a) How far has the wedge moved by the time the cylinder has descended from rest a vertical distance $h$?
(b) Now suppose that the cylinder is free to roll down the wedge without slipping. How far does the wedge move in this case?
(c) In which case does the cylinder reach the bottom faster? How does this depend on the radius of the cylinder?

28. A thin hollow cylinder of radius $R$ and mass $M$ slides across a frictionless floor with speed $V_0$. Initially the cylinder is spinning backward with angular velocity $\omega_0 = 2V_0/R$ as shown in Fig. 1.151. The cylinder passes onto a rough area and continues moving in a straight line. Due to friction, it eventually rolls. What is the final velocity $V_f$?
29. Four rods of length $a$ are arranged as shown. The rods are attached at A, hinged at the masses $M_1$, and the mass $M_2$ is able to move along the vertical line through A. The whole system rotates about this axis with constant angular velocity $\Omega$.
(a) By an elementary analysis of the forces involved, find the equilibrium value of $\theta$ given $\Omega$.
(b) Write down the Lagrangian of the system.
(c) Find the period of small oscillations about the equilibrium value of $\theta$.
(d) If $\Omega$ is increased very gradually from zero to a high value, describe how $\theta$ behaves.

30. A rifle is fired horizontally over the edge of a cliff. The bullet has mass $m$ and muzzle velocity $v_0 = 1km/sec$. The air resistance is given by a force $-mkv^3$, where $k$ is a constant. After 100m the bullet has a velocity of 0.97 km/sec.
(a) What is the velocity of the bullet after it travels a horizontal distance $x = 1km$?
(b) How long has the bullet taken to travel a distance $x = 1km$?
(c) How far has the bullet dropped after 1 km?
You are expected to obtain explicit, simple formulae that will give numerical answer with an accuracy of 1%. You are not required to justify your approximations. For full credit, obtain an explicit numerical answer to part (c), with $g = 9.8m/sec^2$.

31. A rocket of mass $m$ runs out of fuel far from a planet of mass $M$ and radius $R$. If the rocket has a speed $v$ (in the frame of the planet) when it loses power, for what maximum impact parameter $b$ will it strike the planet? Express $b$ in terms of $m$, $M$, $R$, $v$, and the gravitational constant $G$. 
32. The Lagrangian for a relativistic electron in a magnetic field is given by

\[ L = -mc^2 \sqrt{1 - \frac{v^2}{c^2}} - \frac{e}{c} \mathbf{v} \cdot \mathbf{A} \]

When \( \mathbf{A} \) is the vector potential.

(a) Obtain the Hamiltonian.

(b) For the case of a uniform magnetic field (i.e., \( \mathbf{A} = xB_0 \mathbf{\hat{y}} \)) and a particle which is ejected from the origin (i.e., \( x = y = z = 0 \)) with velocity \( \mathbf{v} = \dot{x} \mathbf{v}_0 \), determine the maximum value of \( x \) reached by the particle.

33. A spherical ball of radius \( r \) is inside a vertical circular loop of radius \( (R + r) \) as shown in Fig. 1.166. Consider two cases (i) rolling without sliding (ii) frictionless sliding without rolling.

(a) In each case what minimum velocity \( v_1 \) must the sphere have at the bottom of the loop so as not to fall at the top?

(b) For a 10% smaller \( v_1 \) and the sliding case, where on the loop will falling begin?

34. A cone of height \( h \) and base radius \( R \) is constrained to rotate about its vertical axis, as shown in Fig.1.141. A thin, straight groove is cut in the surface of the cone from apex to base as shown. The cone is set rotating with initial angular velocity \( \omega_0 \) around its axis and a small (point-like) bead of mass \( m \) is released at the top of the frictionless groove and is permitted to slide down under gravity.

Assume that the bead stays in the groove, and that the moment of inertia of the cone about its axis is \( I_0 \).

(a) What is the angular velocity of the cone when the bead reaches the bottom?

(b) Find the speed of the bead in the laboratory just as it leaves the cone.
35. Let $I_1, I_2, I_3$ be the principal moments of inertia (relative to the center of mass) of a rigid body and suppose these moments are all different with $I_1 > I_2 > I_3$.

If the body in free space is set to spin around one of the principal axes, it will continue spinning about that axis. However, we are concerned about the stability. What happens if the initial spin axis is very close to, but not exactly aligned with, a principal axis? Stability implies that the spin axis never wanders far from that principal axis. One finds that the motion is in fact stable for the principal axis. One finds that the motion is in fact stable for the principal axes corresponding to $I_1$ and $I_3$, the largest and the smallest moments of inertia. Explain this analytically using Euler’s equations.

36. Consider the Lagrangian

$$L = \frac{1}{2} m (x^2 - \omega^2 x^2) e^{\gamma t}$$

for the motion of a particle of mass $m$ in one dimension ($x$). The constants $m, r$ and $\omega$ are real and positive.

(a) Find the equation of motion.

(b) Interpret the equation of motion by stating the kinds of force to which the particle is subject.

(c) Find the canonical momentum, and from this construct the Hamiltonian function.

(d) Is the Hamiltonian a constant of the motion? Is the energy conserved? Explain.

(e) For the initial conditions $x(0) = 0$ and $\dot{x}(0) = v_0$, what is $x(t)$ asymptotically as $t \to \infty$?

37. A bowling ball of uniform density is thrown along a horizontal alley with initial velocity $v_0$ in such a way that it initially slides without rolling. The ball has mass $m$ coefficient of static friction $\mu_s$ and coefficient of sliding friction $\mu_d$ with the floor. Ignore the effect of air friction.

Compute the velocity of the ball when it begins to roll without sliding.
38. A railroad flatcar of mass $M$ can roll without friction along a straight horizontal track as shown in Fig.1.86. $N$ men, each of mass $m$, are initially standing on the car which is at rest.

(a) The $N$ men run to one end of the car in unison; their speed relative to the car is $V_r$ just before they jump off (all at the same time). Calculate the velocity of the car after the men have jumped off.

(b) The $N$ men run off the car, one after the other (only one man running at a time) each reaching a speed $V_r$ relative to the car just before jumping off. Find an expression for the final velocity of the car.

(c) In which case, (a) or (b), does the car attain the greater velocity?

![Fig. 1.86.](image)

39. A uniform thin rigid rod of mass $M$ is supported by two rapidly rotating rollers, whose axes are separated by a fixed distance $a$. The rod is initially placed at rest asymmetrically, as shown in Fig.1.114.

(a) Assume that the rollers rotate in opposite directions as shown in the figure. The coefficient of kinetic friction between the bar and the rollers is $\mu$.

Write down the equation of motion of the bar and solve for the displacement $x(t)$ of the center C of the bar from roller 1 assuming $x(0) = x_0$ and $\dot{x}(0) = 0$.

(b) Now consider the case in which the directions of rotation of the rollers are reversed, as shown in Fig.1.115. Calculate the displacement $x(t)$, again assuming $x(0) = x_0$ and $\dot{x}(0) = 0$.

![Fig. 1.114.](image) ![Fig. 1.115.](image)

40. A hemispherical bowl of radius $R$ rotates around a vertical axis with constant angular speed $\Omega$. A particle of mass $M$ moves on the interior surface of the bowl under the influence of gravity (Fig.1.77). In addition, this particle is subjected to a
frictional force $F = -kV_{rel}$; where $k$ is a constant and $V_{rel}$ is the velocity of the particle relative to the bowl.

(a) If the particle is at the bottom of the bowl ($\theta = 0$), it is clearly in equilibrium.

Show that if $\Omega > \sqrt{\frac{g}{R}}$, there is a second equilibrium value of $\theta$ and determine its value.

(b) Suppose the particle is in equilibrium at the bottom of the bowl. To describe the motion of the particle in the vicinity of the equilibrium point, we construct a local inertial Cartesian coordinate system $(x, y, z)$ and neglect the curvature of the bowl except in calculating the gravitational restoring force. Show that for $|x| << R$, $|y| << R$, the particle position satisfies $x = \text{Re}(x_0 e^{i\gamma})$, $y = \text{Re}(y_0 e^{i\gamma})$, where

$$(\lambda^2 + \frac{k\lambda}{M} + \frac{g}{R})^2 + (\frac{k}{M})\Omega^2 = 0$$

(c) Find the angular speed of the bowl, $\Omega_0$, for which the particle’s motion is periodic.

(d) There is a transition from stable to unstable at $\Omega = \Omega_0$. By considering behavior of frequencies in the vicinity of $\Omega_0$, prove that the motion is stable for $\Omega < \Omega_0$ and unstable for $\Omega > \Omega_0$.

41. Consider a collection of point particles of mass $m$ moving in circular orbits about a common center each with the same kinetic energy. If the only force present is the mutual (Newtonian) gravitational force, what is the particle density as a function of radius $r$ from the center in order that the density remains constant in time? (Assume that the density is spherically symmetric.)

42. A long thin uniform bar of mass $M$ and length $L$ is hung from a fixed (assumed frictionless) axis at A as shown in Fig.1.131. The moment of inertia about A is $ML^2/3$. 
(a) An instantaneous horizontal impulse $J$ is delivered at $B$, a distance $a$ below $A$. What is the initial angular velocity of the bar?
(b) In general, as a result of $J$, there will be an impulse $J'$ on the bar from the axis at $A$. What is $J'$?
(c) Where should the impulse $J$ be delivered in order that $J'$ be zero?

![Diagram](image1)

Fig. 1.131.

43. A massless spring of rest length $l_0$ (with no tension) has a point mass $m$ connected to one end and the other end fixed so the spring hangs in the gravity field as shown in Fig. 2.1. The motion of the system is only in one vertical plane.
(a) Write down the Lagrangian.
(b) Find Lagrange’s equations using variables $\theta, \dot{\lambda} = (r - r_0)/r_0$, where $r_0$ is the rest length (hanging with mass $m$). Use $\omega_r^2 = k/m$, $\omega_p^2 = g/r_0$.
(c) Discuss the lowest order approximation to the motion when $\lambda$ and $\theta$ are small with the initial conditions $\theta = 0$, $\dot{\lambda} = 0$, $\lambda = A$, $\dot{\theta} = \omega_p B$ at $t = 0$. $A$ and $B$ are constants.
(d) Discuss the next order approximation to the motion. Under what conditions will the $\lambda$ motion resonate? Can this be realized physically?

![Diagram](image2)

Fig. 2.1.

44. A satellite of mass $m$ moves in a circular orbit of radius $R$ with speed $v$ about the earth. It abruptly absorbs a small mass $\delta m$ which was stationary prior to the collision. Find the change in the total energy of the satellite and, assuming the
new orbit is roughly circular, find the radius of the new orbit.

Two circular metal disks have the same mass $M$ and the same thickness. Disk 1 has a uniform density $\rho_1$, which is less than $\rho_2$, the uniform density disk 2. Which disk, if either, has the larger moment of inertia?

45. A chain with mass/length $= u$ hanging vertically from one end, where an upward force $F$ is applied to it, is lowered onto a table as shown in Fig. 1.103. Find the equation of motion for $h$, the height of the end above the table ($h$ is the length of chain hanging freely).

46. Given that the moment of inertia of a cube about an axis that passes through the center of mass and the center of one face is $I_0$, find the moment of inertia about an axis through the center of mass and one corner of the cube.

47. Consider a particle of mass $m$ moving in a bound orbit with potential $V(r) = -\frac{k}{r}$. Using polar coordinates in the plane of the orbit:

(a) Find $p_r$ and $p_\theta$ as functions of $r, \theta, \dot{r}$ and $\dot{\theta}$. Is either one constant?

(b) Using the virial theorem show that

\[ J_r + J_\theta = \oint \frac{k}{r} \, dr \]

where

\[ J_r = \oint p_r \, dr \]
\[ J_\theta = \oint p_\theta \, d\theta \]

(c) Show that

\[ (J_r + J_\theta) = \sqrt{\frac{-2\pi^2 mk^2}{E}} \]

using
\[
\int_\gamma \frac{dr}{\sqrt{-r^2 + ar - b}} = \pi, \quad r_\pm = \frac{1}{2} (a \pm \sqrt{a^2 - 4b}) .
\]

(d) Using the results of (c) show that the period of the orbit is the same for the \( r \) and \( \theta \) motions, namely,
\[
\tau = \pi k \sqrt{\frac{m}{2E^3}}.
\]

48. Consider a particle of mass \( m \) moving in a plane under a central force
\[
F(r) = -\frac{k}{r^2} + \frac{k'}{r^3}
\]
(assume \( k > 0 \)).

(a) What is the Lagrangian for this system in terms of the polar coordinates \( r, \theta \) and their velocities?

(b) Write down the equations of motion for \( r \) and \( \theta \), and show that the orbital angular momentum \( l \) is a constant of the motion.

(c) Assume that \( l^2 > -mk' \). Find the equation for the orbit, i.e. \( r \) as a function of \( \theta \).

49. Consider the two-body system consisting of (1) a point particle of mass \( m \) and (2) a rotator of finite size and mass \( M \) (see Fig.2.71). This rotator is a rigid body which has uniform density, has an axis of symmetry, and, like the particle of mass \( m \), is free to move. Discuss the motion of this system if the particle is attracted to every element of the rotator by a Coulomb or gravitational force. Include in your discussion answers to the following questions.

(a) How many degrees of freedom does this system have?

(b) What would be a suitable set of coordinates?

(c) What is the Lagrangian (or Hamiltonian)? (Write it down or say how you would try.)

(d) On what coordinates does the interaction between the particle and the rotator depend?

(e) How many constants of motion can you infer, and what are they physically?

(f) What orbits of this system are closely similar to orbits of two point masses? Describe the nature of their (small) difference. What is the nature of the motion of the rotator relative to its center of mass?
50. A rectangle coordinate system with axes $x, y, z$ is rotating relative to an inertial frame with constant angular velocity $\omega$ about the $z$-axis. A particle of mass $m$ moves under a force whose potential is $V(x, y, z)$. Set up the Lagrange equations of motion in the coordinate system $x, y, z$. Show that their equations are the same as those for a particle in a fixed coordinate system acted on by the force $-\nabla V$ and a force derivable from a velocity-dependent potential $U$. Find $U$.

51. A light, uniform U-shaped tube is partially filled with mercury (total mass $M$, mass per unit length $\rho$) as shown in Fig. 2.73. The tube is mounted so that it can rotate about one of the vertical legs. Neglect friction, the mass and moment of inertia of the glass tube, and the moment of inertia of the mercury column on the axis of rotation.
   (a) Calculate the potential energy of the mercury column and describe its possible motion when the tube is not spinning.
   (b) The tube is set in rotation with an initial angular velocity $\omega_0$ with the mercury column at rest vertically with a displacement $z_0$ from equilibrium.

1) Give the Lagrangian for the system.
2) Give the equation of motion.
3) What quantities are conserved in the motion? Give expressions for these quantities.
4) Describe the motion qualitatively as completely as you can.

52. Consider the system of particles $m_1 = m_2$ connected by a rope of length $l$ with $m_2$ constrained to stay on the surface of an upright cone of half-angle $\alpha$ and
$m_1$ hanging freely inside the cone, the rope passing through a hole at the top of the cone as shown in Fig.2.76. Neglect friction.

(a) Give an appropriate generalized coordinate system for the problem.
(b) Write the Lagrangian of the system and the equation of motion for each generalized coordinate.
(c) Write the Hamiltonian for the system.
(d) Express the angular frequency for $m_2$ moving in a circular orbit in terms of the variables of the problem.

53. Two identical discs of mass $M$ and radius $R$ are supported by three identical torsion bars, as shown in Fig.2.8, whose restoring torque is $\tau = -k\theta$ where $k=$given torsion constant for length $l$ and twist angle $\theta$. The discs are free to rotate about the vertical axis of the torsion bars with displacements $\theta_1, \theta_2$ from equilibrium position. Neglect moment of inertia of the torsion bars. For initial conditions $\theta_1(0) = 0$, $\theta_2(0) = 0$, $\dot{\theta}_1(0) = 0$, $\dot{\theta}_2(0) = \Omega = \text{given constant}$, how long does it take for disc 1 to get all the kinetic energy? You may leave this in the form of an implicit function.

54. A block of mass $M$ is rigidly connected to a massless circular track of radius $a$ on a frictionless horizontal table as shown in Fig.2.7. A particle of mass $m$ is confined to move without friction on the circular track which is vertical.

(a) Set up the Lagrangian, using $\theta$ as one coordinate.
(b) Find the equations of motion.
(c) In the limit of small angles, solve that equations of motion for $\theta$ as a function of time.

![Diagram of a simple pendulum](image)

**Fig. 2.7.**

55. The point of support of a simple pendulum moves in a vertical circle of radius $R$ as shown in the figure, with a constant speed $V$. Obtain Lagrange’s equation of motion for the pendulum. (Assume the motion to take place in the vertical plane of the circle).

![Diagram of a pendulum moving in a vertical circle](image)

56. A rigid body rotates freely about its center of mass. There are no torques. Show by means of Euler’s equations that, if all three principal moments of inertia are different then the body will rotate stably about either the axis of greatest moment of inertia or the axis of least moment of inertia, but that rotation about the axis of intermediate moment of inertia is unstable.

57. If a particle is projected vertically upward from a point on the earth’s surface at northern latitude $\lambda$, show that it strikes the ground at a point $4/3\omega \cos \lambda \sqrt{8h^3/g}$ to the west (neglect air resistance and consider only small vertical heights).
58. A thin hoop of radius $R$ and mass $M$ is allowed to oscillate in its own plane with one point of the hoop fixed. Attached to the hoop is a small mass, $m$, which is constrained to move (in a frictionless manner) along the hoop. Consider only small oscillations and show that the eigenfrequencies are 

$$\omega_1 = \sqrt{2} \sqrt{g/R} \quad \omega_2 = \frac{\sqrt{2}}{2} \sqrt{g/R}$$

59. A spring has a length $L$ in equilibrium (lying horizontally on a frictionless table), a mass per unit length $m$, and a spring constant $k$. When hanging vertically from a rigid support with a mass $M$ attached to the free end, what is the extension of the spring? Hint: First, prove that if $y(x)$ is the extension of the point $x$, then the tension at the point $x$ is $kL(dy/dx)$.

60. A bar of length $2L$, mass $m$, slides without friction on a horizontal plane. Its velocity is perpendicular to the axis of the bar. It makes a fully elastic impact (energy conserved) with a fixed peg at a distance $a$ from the center of the bar. Using conversation of energy, momentum and angular momentum, find the final velocity of the center of mass.

61. A particle moves under the action of an attractive central force which is proportional to the displacement from the origin. The Lagrangian in cylindrical coordinates is

$$L = \frac{m}{2} \left( r^2 + r^2 \dot{\theta}^2 \right) - \frac{1}{2} kr^2$$

(a) What are two constants of the motion? (1)
(b) Find the maximum and minimum values of $r$ as functions of the constants of motion. (2)
(c) Under what conditions is the orbit a circle? (1)
(d) Under what conditions is the orbit a straight line? (1)
(e) Write down a formal solution to the problem. (2)
(f) What is the frequency of the motion? (3)

62. A pendulum consists of a weightless rod of length $l$ and bob of mass $m_1$. The support of the pendulum is connected to a mass $m_2$. The mass $m_2$ is connected by a spring (spring constant $= k$) to a rigid wall. The mass $m_2$ sliders on a frictionless, horizontal plane. The pendulum swings in a plane perpendicular to the wall. Find the frequencies of the normal modes and discuss the method of finding the motion corresponding to each of the normal modes.
63. A heavy particle is constrained to move on the inside surface of a smooth spherical shell of inner radius $a$. Gravity is acting vertically downward. Initially the particle is projected with a horizontal velocity $V_0$ from a point which is at a depth $b$ below the center of the sphere. Find the depth $z$ below the center of the sphere when the particle is again moving horizontally. Briefly describe the motion.

64. A particle of mass $m$ moves in a plane under the action of the potential $V(r, \theta) = f(r) + (b/r^2)\sin^2(\theta)$, where $(r, \theta)$ are polar coordinates, $b$ is a constant, and the function $f(r)$ is unspecified. As shown in the figure below, the orbit first crosses the line $\theta = 0$ and latter the line $\theta = \pi/2$. If the angular momentum $\theta = 0 \Rightarrow \theta = \pi/2$ has the value $p_\theta = l_0$ at $\theta = 0$, what is the value at $\theta = \pi/2$?

65. A double pendulum has equal lengths, but the upper mass is much greater than the lower. Obtain the exact Lagrangian for motion in a vertical plane, and then make the approximation of small motion. What are the resonant frequencies of the system? What is the resultant motion if the system initially at rest is subjected at time $t = 0$ to a small impulsive force applied horizontally to the upper mass?
66. Consider two particles of mass \( m_1 \) and \( m_2 \) interacting by an attractive central force. Obtain an integral expression for the orbit in terms of polar coordinates. Use the formula to write an expression in terms of a definite integral for the change in polar angle, \( \Delta \phi \), as \( r \) varies from its minimum value to its maximum value and back to its minimum. For an inverse square force the change in angle should be \( 2\pi \). Assume a potential of the form \( U = -k/r + \delta U \) where \( \delta U \) is a small perturbation and obtain an integral expression for the first order correction to \( \Delta \phi \). What is the correction if \( \delta U = \alpha/r^2 \)?

67. (a) Write the general Hamilton-Jacobi equation for the action \( S \). Assuming that the Hamiltonian is time-independent separate \( S \) into a time dependent part and a time-independent part, \( W \), to derive an equation for \( W \), which is usually called “Hamilton’s characteristic function”.

(b) Write the differential equation for \( W \) for the problem of the three dimensional harmonic oscillator with unequal force constants, \( k_x \), \( k_y \) and \( k_z \). Use the method of action-angle variables to calculate the frequencies of vibration.

68. Two particles of equal mass \( m \) interact according to the potential,

\[
V = 0, r > a \\
V = -V_0, r < a
\]

Initially the particles are separated by some distance \( r > a \), one is at rest and the order has velocity \( V_0 \). Calculate the differential cross section for scattering.

69. Consider a classical treatment of the small oscillations of the atoms about equilibrium in a linear tri-atomic molecule. Two of the masses are equal and at equilibrium are located a distance, \( a \), from the third. Making a reasonable assumption about the functional form of the potential for small motions, obtain the frequencies of vibration. Describe the mode of vibration for each of the eigenfrequencies.

70. A solid cylinder of radius \( b \) has a cylindrical hole of radius \( b/\sqrt{2} \) cut out of it. The hole is centered at a distance \( c \) from the center of the cylinder. This cylinder is at rest on top of a large perfectly rough, fixed cylinder of radius \( R \) as shown. For what values of \( R \) is the equilibrium position shown stable, and what will be the frequency of small oscillations about this equilibrium position?
71. A particle of mass \( m \) is placed in a smooth uniform tube of mass \( M \) and length \( l \). The tube is free to rotate about its center in a vertical plane. The system is started from rest with the tube horizontal and the particle a distance \( r_0 \) from the center of the tube. For what length of the tube will the particle leave the tube when \( \dot{\theta} = \omega \) is a maximum and \( \theta = \theta_m \)? Your answer should be in terms of \( \omega \) and \( \theta_m \).

72. Two electrons interact with each other electrostatically and move in a uniform magnetic field. Using the vector potential \( \vec{A} = (\vec{B} \times \vec{r})/2 \), write down a Lagrangian \( L(r_1, r_2, v_1, v_2) \) which governs the motion of the two electrons. Solve for the motion of the center of mass of the two electrons (Hint: introduce reduced mass coordinates).

73. Two equal point masses \( M \) are connected by a massless rigid rod of length \( 2A \) (a dumbbell) which is constrained to rotate about an axle fixed to the center of the rod at an angle \( \theta \) (Fig.1.134). The center of the rod is at the origin of coordinates, the axle along the \( z \)–axis, and the dumbbell lies in the \( xz \)–plane at
The angular velocity $\omega$ is a constant in time and is directed along the $z$-axis.

(a) Calculate all elements of the inertia tensor. (Be sure to specify the coordinate system you use).
(b) Using the elements just calculated, find the angular momentum of the dumbbell in the laboratory frame as a function of time.
(c) Using the equation $L = r \times P$, calculate the angular momentum and show that it is equal to the answer for part (b).
(d) Calculate the torque on the axle as a function of time.
(e) Calculate the kinetic energy of the dumbbell.

74. A sphere of mass $m$, radius $a$, and moment of inertia $\frac{2}{5}ma^2$ rolls without slipping from its initial position at rest atop a fixed cylinder of radius $b$ (see Fig1.162)
(a) Determine the angle $\theta_{\text{max}}$ at which the sphere leaves the cylinder.
(b) What are the components of the velocity of the sphere’s center at the instant it leaves the cylinder?

75. A disk of mass $M$ and radius $R$ slides without friction on a horizontal surface. Another disk of mass $m$ and radius $r$ is pinned through its center to a
point off the center of the first disk by a distance $b$, so that it can rotate without friction on the first disk as shown in Fig. 2.2. Describe the motion and identify its constants.

![Fig. 2.2.](image)

76. A thin disk of radius $R$ and mass $M$ lying in the $xy$-plane has a point mass $m = 5M/4$ attached on its edge (as shown in Fig. 1.109). The moment of inertia of the disk about its center of mass is (the $z$-axis is out of the paper).

$$I = \frac{MR^2}{4} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

![Fig. 1.109.](image)

(a) Find the moment of inertia tensor of the combination of disk and point mass about point A in the coordinate system shown.

(b) Find the principal moments and the principal axes about point A.

(c) The disk is constrained to rotate about the $y$-axis with angular velocity $\omega$ by pivots at A and B. Describe the angular momentum about A as a function of time and find the vector force applied at B (ignore gravity).

77. Consider a classical system of point masses $m_i$ with position vectors $r_i$, each experiencing a net applied force $F_i$.

(a) Consider the quantity $\sum_i m_i \dot{r}_i \cdot r_i$, assumed to remain finite at all times, and
prove the virial theorem.

\[ T = -\frac{1}{2} \sum_i F_i \cdot r_i \],

Where \( T \) is the total kinetic energy of the system and the bar denotes time average.

(b) In the case of a single particle acted on by a central inverse-square law force, show that

\[ T = -\frac{V}{2} \],

where \( V \) is the potential energy.

78. A pendulum consists of a uniform rigid rod of length \( L \), mass \( M \), a bug of mass \( M/3 \) which can crawl along the rod. The rod is pivoted at one end and swings in a vertical plane. Initially the bug is at the pivot-end of the rod, which is at rest at an angle \( \theta_0 \) (\( \theta_0 << 1 \) rad) from the vertical as shown in Fig. 1.143, is released. For \( t > 0 \) the bug crawls slowly with constant speed \( V \) along the rod towards the bottom end of the rod.

(a) Find the frequency \( \omega \) of the swing of the pendulum when the bug has crawled a distance \( l \) along the rod.

(b) Find the amplitude of the swing of the pendulum when the bug has crawled to the bottom end of the rod (\( l = L \)).

(c) How slowly must the bug crawl in order that your answer for part (a) and (b) be valid?

79. Is there any value of the parameter \( \alpha \) for which the following transformation is canonical [Hint: \((d/dx)\tan^{-1}(x) = (1+x^2)^{-1}\)]

\[ P = \frac{q^2 + p^2}{\alpha}, \quad Q = \tan^{-1} \frac{p}{q} \]?
80. Consider an electron which moves in a uniform magnetic field and an electrostatic field [i.e. \( B = \times A \) and \( E = -\phi \), where \( A = +xB\hat{y} \) and \( \phi = \phi(x, y, z) \)]. When \( \phi(x, y, z) \) varies by only a small amount on the scale of the electron Larmor radius, the transverse kinetic energy [i.e. \( (m\hat{x}^2 + m\hat{y}^2) / 2 \)] is nearly constant and may be neglected in the Lagrangian. The location and z-velocity of the electron are determined approximately by the Lagrangian

\[
L = \frac{m\hat{z}^2}{2} - \frac{e}{c} Bx\hat{y} + e\phi(x, y, z).
\]

Identify coordinates and conjugate momenta and write down the Hamiltonian.

81. One of the following two Hamiltonians is integrable

\[
H_1 = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + V \sin(q_1 - q_2)
\]

\[
H_2 = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + V \sin(q_1)\sin(q_2),
\]

where \( V \) is an arbitrarily large constant. State which Hamiltonian is integrable and prove your answer.

82. An electron moves in the uniform electric and magnetic fields \( E = E\hat{x} \) and \( B = B\hat{z} \). For a judicious choice of the vector and scalar potentials, solve the Hamilton-Jacobi equation which governs the motion of the electron.

83. (a) Is the following transformation canonical

\[
Q_1 = q_2, \quad P_1 = p_1
\]

\[
Q_2 = q_1, \quad P_2 = p_2?
\]

(b) Given that \((x, y, z, p_x, p_y, p_z)\) are canonical variables and that

\[
\frac{1}{r} + (q / c)A(r),
\]

evaluate \([v_x, v_y]\).

84. (a) A particle moves under the action of a central force. Is it possible for

\[
L_x = yp_z - zp_y, \quad L_y = zp_x - xp_z, \quad \text{and} \quad L_z = xp_y - yp_x
\]

to be the new momenta in a solution of the Hamiltonian-Jacobi equation for the particle? Explain your answer.

(b) The Hamiltonian for a system of particles is given by
\[ H = \sum_{j} \left| \frac{p_{j}}{2m} + \sum_{j} v(x_{j}^{2} + y_{j}^{2}) + \sum_{\nu < j} e^2 / |\xi_{\nu} - \xi_{j}| \right|. \]

Obtain two functions which generate infinitesimal transformations under which \( H \) is invariant.

85. A bead of mass \( m \) slides freely on a wire hoop of radius \( r \), as shown in the figure below. Gravity is directed vertically down, and the hoop rotates with constant angular velocity \( \omega \) about a vertical axle which bisects the hoop. Determine the condition that there exist a stable equilibrium for \( \theta \neq 0 \).

86. Three masses \((m_1, m_2, m_3)\), forming the corners of an equilateral triangle, attract each other according to Newton’s Law. Determine the rotational motion which will leave the relative position of these masses unchanged.

87. A mass \( m \) moves in a circular orbit of radius \( r_0 \) under the influence of a central force whose potential is \(-kr^n\). Show that the circular orbit is stable under small oscillations (that is, the mass will oscillate about the circular orbit) if \( n < 2 \).

88. Two particles move about each other in circular orbits under the influence of gravitational forces, with a period \( \tau \). The motion is suddenly stopped at a given instant of time, and the particles are then released and allowed to fall into each other. Prove that they collide after a time \( \tau / (4\sqrt{2}) \).
89. A uniform thin rigid rod of weight $W$ is supported horizontally by two vertical props at its ends. At $t = 0$ one of these supported is kicked out. Find the force on the other support immediately thereafter.

90. Three identical cylinders with parallel axes are in contact with each other on a rough plane, with two cylinders lying on the plane and the third resting on top of them, as in the figure. What is the minimum angle which the direction of the force acting between the cylinders and the plane makes with the vertical?

91. A yo-yo rests on a level surface. A gentle horizontal pull (see figure) is exerted on the cord so that the yo-yo rolls without slipping. Which way does it roll and why?

92. The surface of a sphere is vibrating slowly in such a way that the principal moments of inertia are harmonic functions of time:

\[
I_{zz} = \frac{2mr^2}{5} (1 + \varepsilon \cos \omega t), \quad I_{xx} = I_{yy} = \frac{2mr^2}{5} (1 - \varepsilon \cos \omega t)
\]

where $\varepsilon << 1$. The sphere is simultaneously rotating with angular velocity $\Omega(t)$. Show that the $z$-component of $\Omega$ remains approximately constant. Show also that $\Omega(t)$ precesses around $z$ with a precession frequency $\omega_p = (3\varepsilon \Omega_z / 2) \cos \omega t$ provided $\Omega_z >> \omega$.

93. Three rigid spheres are connected by light, flexible rods with relative masses as shown below:

Describe all the normal modes of the system and state whatever you can about the relative frequencies.

94. A rigid uniform bar of mass $M$ and length $L$ is supported in equilibrium in a
horizontal position by two massless springs attached one at each end.

The springs have the same force constant $k$. The motion of the center of gravity is constrained to move parallel to the vertical $X$ axis. Find the normal modes and frequencies of vibration of the system, if the motion is constrained to the $XZ$-plane.

95. A particle of mass $M$ hangs from one end of a uniform string of mass $m$ and length $L$; the other end of the spring is fixed. The particle is given a small lateral displacement $\delta$ and released from rest. Set up the differential equations and boundary conditions to determine the motion of string and particle. Set up a transcendental equation that determines the natural frequencies, and solve the equation for the case $m < M$.

96. Set up a variational principle for the frequency $\omega$ of a membrane with surface tension $T$, of mass $\sigma$ per unit area, and with fixed edges; that is, find an integral over the area of the membrane, of which the extreme value is the frequency of the membrane.

97. The free surface of a liquid is one of constant pressure. If an incompressible fluid is placed in a cylinder vessel and the whole rotated with constant angular velocity $\omega$, show that the free surface becomes a paraboloid of revolution.

98. A positron (energy $E_+$, momentum $p_+$) and an electron (energy $E_-$, momentum $p_-$) are produced in a pair-creation process.
   (a) What is the velocity of the frame in which the pair has zero momentum (barycentric frame)?
   (b) Deduce the energy either particle has in this frame
   (c) Give an expression for the magnitude of the relative velocity between the particles i.e. the velocity of one particle as seen by an observer attached to the other.

99. A fast (extremely relativistic) electron enters a condenser at an angle $\alpha$ as shown in the sketch. $V$ is the voltage across the condenser and $d$ is the distance
between plates. Give an equation for the path of the electron in the condenser.